Math 206B Lecture 5 Notes

Daniel Raban

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1 Tableau

1.1 Young diagrams and tableau

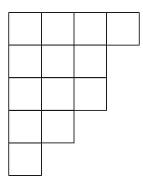
We want to work toward the following theorem:

Theorem 1.1. $M^\mu = \bigoplus_{\lambda \geq \mu} m_{\mu,\lambda} S^\lambda$, *where*

$$m_{\mu,\lambda} := \langle M^\mu, S^\lambda \rangle = \frac{1}{n!} \sum_{\nu} z_\nu \chi_{M^\mu}[\nu] \chi_{S^\lambda}[\nu].$$

Definition 1.1. Given a partition λ, the **Young diagram** of λ is the partition expressed as stacked rows of boxes.

Example 1.1. Take $\lambda = (4, 3, 3, 2, 1)$. The Young diagram of λ is



Definition 1.2. A **standard Young tableau**[1] is a Young diagram where we fill in the boxes with the numbers 1 to n, according to the rule that the numbers have to be increasing going to the right and going down.

Example 1.2. Here is a Young tableau:

1	2	3	7
4	5	10	
6	8		
9	12		
11			

[1] The plural of tableau is tableaux.

1.2 Poly-tabloids and their relation to irreducible representations of S_n

Definition 1.3. A **tabloid** is a Young diagram filled with the numbers $\{1, \ldots, n\}$. A **standard tabloid** is the same thing, except you need that numbers are increasing going to the right.

Example 1.3. Here is a tabloid:

$$\begin{array}{|c|c|c|c|c|c|}\hline 3 & 5 & 6 & 8 & 10 & 11 \\ \hline 1 & 2 & 7 & 13 \\ \cline{1-4} 4 & 12 \\ \cline{1-2}\end{array}$$

Observe that λ-tabloids are in bijection with λ-flags.

Definition 1.4. A **poly-tabloid** is an equivalence class of tabloids under the action of $S_{\lambda_1} \times S_{\lambda_2} \times \cdots$.

Let $R^\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots$ be the group of row permutations on λ-tabloids, and let $C^\lambda = S_{\lambda'_1} \times S_{\lambda'_2} \times \cdots$ be the group of column permutations on λ-tabloids.

Let

$$\mathcal{X}_\lambda = \sum_{\sigma \in C^\lambda} \operatorname{sign}(\sigma)\sigma$$

be an element of the group algebra $\mathbb{C}[S_n]$.

Example 1.4. Let $\lambda = (3\,2)$.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$$

Then $\mathcal{X}_\lambda = (1 - (1\,4))(1 - (2\,5))$. where 1 is the identity in the group algebra.

We can also think of M^λ as S_n acting on $\mathbb{C}\langle \lambda - \text{poly-tabloids}\rangle$. Define $e_t := \mathcal{X}_t\{t\}$, where $\mathcal{X}_t$ is the projection of $\mathcal{X}$ onto $\mathbb{C}\langle\{t\}\rangle$.

Example 1.5. Suppose

$$t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$$

Then

$$e_t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 4 & 5 & 2 \\ \hline 3 & 1 \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 3 & 5 & 2 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$$

Theorem 1.2. *The S_n action on e_t for t a λ-tabloid is an irreducible representation S^λ.*

What is the idea here? Start with λ, and construct a λ-tabloid t. Now $\mathcal{X}_t$ is the element of $\mathbb{C}[S_n]$ corresponding to t. Then $e_t = \mathcal{X}_t\{t\}$ is a linear combination of poly-tabloids. If we let $W_\lambda = \mathbb{C}\langle\{t\}\rangle$ be the linear span of poly-tabloids (M^λ acts on W_λ), then $\sigma \cdot e_t \in E_\lambda$. Now we can think of $\mathbb{C}\langle\{\sigma e_t : \sigma \in S_n\}\rangle$. The claim is that this is isomorphic to S^λ. That is, if we define S^λ like this, then the claim is that these are all irreducible and distinct.